FUNCTIONAL EQUATIONS

Ozgur Kircak

April 8, 2011

Contents

Chapter 1

FUNCTIONS BASICS

Definition 1. *A function is a rule, relation, correspondence from a set, called domain, to another set, called range, where each element in the domain is mapped to exactly one element from the range.*

Definition 2. *A function f is even, if $\forall x \in D_f$ we have $f(-x) = f(x)$.*

Definition 3. *A function f is odd, if $\forall x \in D_f$ we have $f(-x) = -f(x)$.*

Definition 4. *A function f is injective or 1-1, if $\forall x_1, x_2 \in D_f$ we have $x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2)$.*

Definition 5. *A function f is surjective or onto, if $\forall y \in E_f$, $\exists x \in D_f$ such that $f(x) = y$.*

Definition 6. *A function f is increasing, if $\forall x_1, x_2 \in D_f$ we have $x_1 < x_2 \Longrightarrow f(x_1) < f(x_2)$.*

Definition 7. *A function f is decreasing, if $\forall x_1, x_2 \in D_f$ we have $x_1 < x_2 \Longrightarrow f(x_1) > f(x_2)$.*

1.1 EXERCISES

1. Determine if the following functions are increasing or decreasing on their domains.

   (a) $f(x) = 2x + 3$

   (b) $f(x) = 5 - 3x$

   (c) $f(x) = \frac{2x+3}{3x}$

   (d) $f(x) = 2^x + x$

   (e) $f(x) = \frac{12x+5}{4x+1}$

   (f) $f(x) = \frac{ax+b}{cx+d}$ where $ad - bc > 0$.

2. Determine if each of the following function is even or odd.

   (a) $f(x) = 2x^3 + x$

   (b) $f(x) = 2x^4 + 3x^2 + 7$

   (c) $f(x) = \frac{2|x|+1}{x^4-1}$

   (d) $f(x) = \begin{cases} 2x+1 & \text{if } x < 0 \\ -2x-1 & \text{if } x \geq 0 \end{cases}$

   (e) $f(x) = \begin{cases} 3x^2+1 & \text{if } x < 0 \\ -3x^2-1 & \text{if } x \geq 0 \end{cases}$

3. Given that $f(x) = \begin{cases} x-1 & \text{if } x > 3 \\ x^2-2x-3 & \text{if } 1 < x \leq 3 \\ x+4 & \text{if } x \leq 1 \end{cases}$. Find $\underbrace{f(f(...f(0)...))}_{2009 times}$.

4. Let $f(2x+1) = 3x+2$. Find $f(2), f(-1)$ and $f(a)$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ and $f(x + \frac{1}{x}) = x^2 + \frac{1}{x^2}$ for $x \in \mathbb{R}$. Find $f(3), f(-5)$ and $f(a)$ where $a \notin (-2, 2)$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ and $f(x - \frac{1}{x}) = x^3 - \frac{1}{x^3}$. Find $f(x)$.

7. Given that $f(x) + f(x+1) = 2x + 3$ and $f(1) = 2$ find $f(99)$.

8. Let $f : \mathbb{R} \to \mathbb{R}$ and $f(x) + 2f(-x) = 2x + 3$. Find $f(5), f(a)$.

9. Let $f : \mathbb{R} \to \mathbb{R}$ and $f(x) + 3f(\frac{1}{x}) = 3x + 2$. Find $f(2), f(a)$.

10. Let $f : \mathbb{R} \to \mathbb{R}$ and $f(x) + 2f(1-x) = 2x + 3$. Find $f(3), f(a)$.

11. Let $f : \mathbb{R} \backslash \left\{\frac{1}{2}\right\} \to \mathbb{R}$ such that $f(x) + 2f(\frac{x+2}{2x-1}) = x$. Find $f(1), f(a)$.

12. Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Compute $f(2009), f(\frac{1}{2009})$ if $f(1) = 3$.

13. Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y) + 2xy$ for all $x, y \in \mathbb{R}$. Compute $f(n), f(\frac{1}{n})$ and $f(\frac{m}{n})$ where $m, n \in \mathbb{N}$ if $f(1) = 1$.

14. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $f(f(x)) = x^2$. Prove that $x^2 < f(x) < x$ for all $x \in (0, 1)$. Give an example of such a function.

15. Solve the equation $\sqrt{\underbrace{1 + \sqrt{1 + ... + \sqrt{1 + x}}}_{2009 times}} = x$.

16. Prove that the sum/product of two increasing functions is also an increasing function.

17. Prove that the sum/product of two even functions is an even function.

18. Prove that the sum of two odd functions is an odd function.

19. Prove that an increasing function is injective.

20. Let f be an increasing function with $f^{(n)}(x) = x$ for all $x \in D_f$. Prove that $f(x) = x$.

21. Solve the equation $x + x^3 + x^5 = x^2 + x^6 + x^{10}$.

22. Solve the equation $2^x - x^2 = 2^{x^2} - x$.

23. Solve the equation $\sqrt{x-3} + \sqrt{21-x} = x^2 - 24x + 150$.

24. Let $f : \mathbb{R} \to \mathbb{R}$ such that $2f(x) + f(1 - x) = x^2$ for all $x \in \mathbb{R}$. Find $f(x)$.

25. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $af(x) + f(\frac{1}{x}) = ax$ where $x \neq 0$, $a \neq \pm 1$.

26. Given that $f(x) = \frac{4^x}{4^x+2}$. Compute the sum

$$f(0) + f(\frac{1}{2009}) + f(\frac{2}{2009}) + ... + f(\frac{2008}{2009}) + f(1).$$

Chapter 2

CAUCHY FUNCTIONAL EQUATION

A functional equation is an equation whose variables range over functions. Thus, to solve a functional equation means to find all functions satisfying the equation. One of the most basic functional equation is

$$f(x+y) = f(x) + f(y)$$

which is called Cauchy functional equation. It is not that difficult to see that any function of the form $f(x) = cx$ satisfies the equation. But the problem is to find all the solutions. If the domain of $f(x)$ is the set of rational numbers, then $f(x) = cx$ is the only solution. But if we extend the domain to reals, there are many noncontinuous solutions. We will give this as a theorem.

Theorem 1. *Let $f : \mathbb{Q} \to \mathbb{Q}$ satisfy $f(x+y) = f(x)+f(y)$ then $f(x) = cx$ where $c = f(1)$.*

Proof. $x = y = 0 \implies f(0+0) = f(0) + f(0) \implies f(0) = 0$.
By the trivial induction, $f(nx) = nf(x)$ (*) for all $n \in \mathbb{N}$ and for all $x \in \mathbb{Q}$.
$x = -y \implies f(x-x) = f(x)+f(-x)$ since $f(0) = 0$ we get $f(x) = -f(-x)$ for all $x \in \mathbb{Q}$.(f is odd.)
Replacing $x = \frac{1}{n}$ in (*), we get that $f(\frac{1}{n}) = \frac{f(1)}{n}$ for all $n \in \mathbb{N}$.
Finally $x = \frac{1}{m}$ in (*) gives $f(\frac{n}{m}) = (\frac{n}{m})f(1)$ for all $n, m \in \mathbb{N}$.
So if we let $f(1) = c$, we have shown that $f(x) = cx$ for all $x \in \mathbb{Q}^+$.
Since f is odd, we have $f(-x) = -f(x) = -cx = c(-x)$.
So for all $x \in \mathbb{Q}$, $f(x) = cx$. □

To extend the problem to the set of real numbers we need some extra conditions such as monotoniticity, continuity.

Theorem 2. *Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x+y) = f(x)+f(y)$ then $f(x) = cx$ where $c = f(1)$ if one of the following conditions is satisfied:*
(i) f is monotone increasing (decreasing)
(ii) f is continuous.

Proof. (i) By Theorem 1, we know that $f(x) = cx$ for all $x \in \mathbb{Q}$. Also by density theorem, for all $x \in \mathbb{R}$, $\exists a_n, b_n \in \mathbb{Q}$ such that $a_n \le x \le b_n$ and $\lim a_n = \lim b_n = x$. Since f is monotone increasing (or decreasing)

$$a_n \le x \le b_n \Longrightarrow f(a_n) \le f(x) \le f(b_n) \text{ or } f(a_n) \ge f(x) \ge f(b_n)$$

if f is monotone decreasing.
Since $a_n, b_n \in \mathbb{Q}$, $f(a_n) = ca_n$ and $f(b_n) = cb_n$.
So $ca_n \le f(x) \le cb_n$ or $ca_n \ge f(x) \ge cb_n$. Taking limits,in either case, the sandwich theorem gives $f(x) = cx$.
(ii) We know that if f is continuous, then $\lim f(a_n) = f(\lim a_n)$.
By density theorem again, for all $x \in \mathbb{R}$, $\exists a_n \in \mathbb{Q}$ such that $\lim a_n = x$.
Hence, $f(x) = f(\lim a_n) = \lim f(a_n) = \lim ca_n = cx$. □

Using Induction To Solve Functional Equations

What we needed to prove Cauchy functional equation is mathematical induction. In this part, we will have a look at some functional equations which can be solved by using mathematical induction.
Remember that, the domain in the Cauchy functional equation is $\mathbb{Q}$. This is a very big hint. When the domain of the function is $\mathbb{Q}$, most probably our tool is induction.

Example 1. *Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that*

$$f(x + y) = f(x) + f(y) + 2xy$$

for all $x, y \in \mathbb{Q}$.

Solution 1. $x = y = 0 \Longrightarrow f(0) = 0$
$x = -y \Longrightarrow f(-x) = -f(x) + 2x^2$ *(1)*
$x = y \Longrightarrow f(2x) = 2f(x) + 2x^2$
$y = 2x \Longrightarrow f(3x) = f(x) + f(2x) + 4x^2 = 3f(x) + 6x^2$
It is easy to see that $f(nx) = nf(x) + n(n-1)x^2$ (*) and to prove it by induction.
It is obviously true for $n = 1$. Assume that it is true for $n = k$, we will show that it is true for $n = k + 1$.

$y = kx \Longrightarrow f(x + kx) = f(x) + f(kx) + 2x \cdot kx$

$$\begin{aligned} f((k+1)x) =& f(x) + f(kx) + 2kx^2 \\ =& f(x) + kf(x) + (k^2 - k)x^2 + 2kx^2 \\ =& (k+1)f(x) + (k^2 + k)x^2 \\ =& (k+1)f(x) + (k+1)(k)x^2 \end{aligned}$$

Hence, (*) is true for all $n \in \mathbb{N}$, for all $x \in \mathbb{Q}$.
Replacing $x = \frac{1}{n}$ in (*) we get that

$$f(\frac{1}{n}) = \frac{c}{n} + \frac{1}{n^2} \qquad (**)$$

where $c = f(1) - 1$.
$x = \frac{1}{m}$ in (*) gives

$$\begin{aligned} f(\frac{n}{m}) =& nf(\frac{1}{m}) + (n^2 - n)\frac{1}{m^2} \\ =& n(\frac{c}{m} + \frac{1}{m^2}) + \frac{n^2 - n}{m^2} \quad \text{by (**)} \\ =& c(\frac{n}{m}) + (\frac{n}{m})^2 \end{aligned}$$

So, for all $x \in \mathbb{Q}^+$, $f(x) = cx + x^2$.
Now let $x > 0$. By using (1) we get:

$$f(-x) = -f(x) + 2x^2 = -cx - x^2 + 2x^2 = -cx + x^2 = c(-x) + 2(-x)^2.$$

Hence, for all $x \in \mathbb{Q}$, $f(x) = cx + x^2$. And that is obviously a solution to the equation.

This problem can be solved in a more elegant way. The first solution was to practice induction and to be able to appreciate the second solution.

Solution 2. *Observe that $f(x) = x^2$ is a solution to the given equation.*
Now let $f(x)$ be the solution of the eqution. Define $g(x) = f(x) - x^2$.
We see that $g(x + y) = g(x) + g(y)$.
So by Thm.1, we get that $g(x) = cx \Longrightarrow f(x) = cx + x^2$.

2.1 EXERCISES

27. Find all functions $f : \mathbb{Q} \to \mathbb{R}$ such that $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{Q}$.

28. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that $f(x + y) = f(x) + f(y) + xy$ for all $x, y \in \mathbb{Q}$.

29. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that
$$f(x + y) = f(x) + f(y) + xy(x + y)$$
for all $x, y \in \mathbb{Q}$.

30. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that $f(1) = 2$ and
$$f(xy) = f(x)f(y) - f(x + y) + 1$$
for all $x, y \in \mathbb{Q}$.

31. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{Q}$:
$$f(\sqrt{x^2 + y^2}) = f(x)f(y).$$

32. Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ satisfying the both conditions:
    (a) $f(x + 1) = f(x) + 1$
    (b) $f(x)^2 = f(x^2)$ for all $x \in \mathbb{Q}^+$.

33. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that $f(x^2 + y) = xf(x) + f(y)$ for all $x, y \in \mathbb{Q}$.

34. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that
$$f(x - y) + f(x + y) = 2f(x)$$
for all $x, y \in \mathbb{Q}$.

35. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the both conditions:
    (a) $f(xy) = f(x)f(y)$
    (b) $f(x + y) = f(x) + f(y) + 2xy$

    for all $x, y \in \mathbb{R}$.

36. (Estonia 2007/5)Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$:
$$f(x + f(y)) = y + f(x + 1).$$

37. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(f(m + n) + f(m - n)) = 8m$ for all $m \in \mathbb{N}$, $n \in \mathbb{N}_0$ with $m > n$.

38. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that
$$f(x + y + z) = f(x) + f(y) + f(z) + 3(x + y)(y + z)(z + x)$$
for all $x, y, z \in \mathbb{Q}$.

Chapter 3

INJECTIVITY-SURJECTIVITY

Example 2. *Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that*

$$f(x) + f(x + f(y)) = 2x + y$$

for all $x, y \in \mathbb{R}$.

Solution 1. Replacing $x = 0$ in the original equation we get

$$f(0) + f(f(y)) = y.$$

So,

$$f(a) = f(b) \Longrightarrow f(f(a)) = f(f(b)) \Longrightarrow a - f(0) = b - f(0) \Longrightarrow a = b$$

Which means that, f is injective.

$$x = y = 0 \Longrightarrow f(0) + f(f(0)) = 0 \Longrightarrow f(0) = -f(f(0)) \qquad (1)$$

$$x = f(0), y = -2f(0) \Longrightarrow f(f(0)) + f\Big(f(0) + f(-2f(0))\Big) = 0 \qquad (2)$$

(1) and (2) $\Longrightarrow f\Big(f(0)+f(-2f(0))\Big) = f(0) \Longrightarrow f(0)+f(-2f(0)) = 0$ due to injectivity

$$(1) \Longrightarrow f(-2f(0)) = f(f(0)) \Longrightarrow -2f(0) = f(0) \Longrightarrow f(0) = 0$$

by injectivity again.
Finally, putting $y = 0$ in the original equation, we see that $f(x) = x$ for all $x \in \mathbb{R}$, which satisfies the equation.

Solution 2. Replacing $x = 0$ in the original equation we get

$$f(0) + f(f(y)) = y.$$

So, f is surjective. That is, for all $y \in \mathbb{R}, \exists x$ (that $x = f(y + f(0))$) such that $f(x) = y$. Hence, there is x_0 such that $f(x_0) = 0$.
Replacing $y = x_0$ in the original equation we get $f(x) = x + \frac{x_0}{2}$. And the original equation forces x_0 to be 0.
So, $f(x) = x$ for all $x \in \mathbb{R}$.

Example 3. ***(BMO 2000/1)*** *Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that*

$$f\Big(xf(x) + f(y)\Big) = f(x)^2 + y$$

for all $x, y \in \mathbb{R}$.

Solution. $x = 0 \Longrightarrow f(f(y)) = f(0)^2 + y$ which means that f is surjective, since for all $y \in \mathbb{R}, \exists x$ (that $x = f(y - f(0)^2)$) such that $f(x) = y$.
So, there is x_0 such that $f(x_0) = 0$.
$x = x_0 \Longrightarrow f(f(y)) = y$ for all $y \in \mathbb{R}$. (1)
$y = x_0 \Longrightarrow f(xf(x)) = f(x)^2 + x_0$ (2)
$x := f(x)$ in (2) $\Longrightarrow f\Big(f(x)f(f(x))\Big) = f(f(x))^2 + x_0$.
By using (1) we get that $f\Big(f(x)x\Big) = x^2 + x_0$. (3)
Comparing (2) and (3) we have $f(x)^2 = x^2$. And that implies $f(0) = 0$.
Due to (1) we have injectivity. So, $f(0) = 0 = f(x_0) \Longrightarrow x_0 = 0$. That is, $f(x) = 0 \Longleftrightarrow x = 0$.

$$f(x)^2 = x^2 \Longrightarrow f(x) = \pm x \text{ for each } x.$$

Namely,

$$f(x) = \begin{cases} x & \text{if } x \in A \\ -x & \text{if } x \notin A \end{cases}$$

We will show that, either $A = \{0\}$ or $A = \mathbb{R}$.
Assume contrary, i.e $\exists a, b \neq 0$ s.t $f(a) = a$ and $f(b) = -b$.
$x = a, y = b$ in the original equation gives $f(a^2 - b) = a^2 + b$.
Since, $f(x) = \pm x$ for each x,
either $a^2 - b = a^2 + b$ or $-(a^2 - b) = a^2 + b$.
$a^2 - b = a^2 + b \Longrightarrow b = 0$ and $-(a^2 - b) = a^2 + b \Longrightarrow a = 0$.
Hence, contradiction.
Therefore, for all $x \in \mathbb{R}$, $f(x) = x$ or for all $x \in \mathbb{R}$, $f(x) = -x$.
Obviously, both functions satisfy the equation.

3.1 EXERCISES

39. Determine all function $f : \mathbb{R} \to \mathbb{R}$ such that $\forall x, y \in \mathbb{R}$:

$$f\big(xf(x) + f(y)\big) = f(x^2) + y.$$

40. Find all function $f : \mathbb{R} \to \mathbb{R}$ such that $\forall x, y \in \mathbb{R}$:

$$f\big(xf(x) + f(y)\big) = \big(f(x)\big)^2 + y.$$

41. Find all function $f : \mathbb{R} \to \mathbb{R}$ such that $\forall x, y \in \mathbb{R}$:

$$f\big(x^2 + f(y)\big) = xf(x) + y.$$

Chapter 4

MIXED PROBLEMS

42. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(x-y) + f(x+y) = 2f(x)$$
for all $x, y \in \mathbb{R}$ and $f(x) > f(0)$ for all $x > 0$.

43. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(x^2 + y) = xf(x) + f(y)$$
for all $x, y \in \mathbb{R}$.

44. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$(x-y)f(x+y) - (x+y)f(x-y) = 4xy(x^2 - y^2)$$
for all $x, y \in \mathbb{R}$.

45. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$xf(y) + yf(x) = (x+y)f(x)f(y)$$
for all $x, y \in \mathbb{R}$.

46. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$xf(y) - yf(x) = (x-y)f(x+y)$$
for all $x, y \in \mathbb{R}$.

47. (Austria 2001/4)Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f((f(x))^2 + f(y)) = xf(x) + y$$
for all $x, y \in \mathbb{R}$.

48. Determine all continuous function $f:\mathbb{R}\to\mathbb{R}$ such that $\forall x,y\in\mathbb{R}$:

$$f\big(x^2f(x)+f(y)\big)=\big(f(x)\big)^3+y.$$

49. Find all functions $f:\mathbb{N}\to\mathbb{N}$ such that

$$f(f(m)+f(n))=m+n$$

for all $m,n\in\mathbb{N}$.

50. (Macedonia 2007/4)Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x^3+y^3)=x^2f(x)+yf(y^2)$$

for all $x,y\in\mathbb{R}$.

51. (Macedonia 2006/2)Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x+y^2+z)=f(f(x))+yf(y)+f(z)$$

for all $x,y,z\in\mathbb{R}$.

52. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(x^3+y)=x^2f(x)+f(y)$ for all $x,y\in\mathbb{R}$.

53. (IMO 87/4)Does there exist a function $f:\mathbb{N}_0\to\mathbb{N}_0$ such that

$$f(f(n))=n+1987$$

for all $n\in\mathbb{N}_0$?

54. (IMO 99/6)Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x-f(y))=f(f(y))+xf(y)+f(x)-1$$

for all $x,y\in\mathbb{R}$.

55. (Iran 99)Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(f(x)+y)=f(x^2-y)+4f(x)y$$

for all $x,y\in\mathbb{R}$.

56. (BMO 2007/2)Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(f(x)+y)=f(f(x)-y)+4f(x)y$$

for all $x,y\in\mathbb{R}$.

57. Find all functions $f:\mathbb{R}\to\mathbb{R}$ that satisfy

$$f(x)f(y)=f(x-y)$$

for all $x,y\in\mathbb{R}$ and $f(2009)=1$.

58. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$f(x + f(y)) = f(x) + y$$

for all $x, y \in \mathbb{Q}$.

59. (BMO 87/1) Let a be a real number and $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(0) = \frac{1}{2}$ and

$$f(x + y) = f(x)f(a - y) + f(y)f(a - x)$$

for all $x, y \in \mathbb{R}$. Prove that f is constant.

60. (China 96) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that for all $x, y \in \mathbb{R}$,

$$f(x^3 + y^3) = (x + y)(f(x)^2 - f(x)f(y) + f(y)^2).$$

Prove that for all $x \in \mathbb{R}$, $f(1996x) = 1996f(x)$.

61. Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ satisfying

    (a) $f(x + 1) = f(x) + 1$

    (b) $f(x)^3 = f(x^3)$ for all $x \in \mathbb{Q}^+$.

62. Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that $f(n) + f(f(n)) = 6n$ for all $n \in \mathbb{N}$. Find $f(n)$.

63. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

for all $x, y \in \mathbb{R}$.

64. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

$$f(x)f(y) - f(x + y) = x + y$$

for all $x, y \in \mathbb{R}$.

65. (Italy TST 2006/3) Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ that satisfy

$$f(m - n + f(n)) = f(m) + f(n)$$

for all $m, n \in \mathbb{Z}$.

66. (Italy TST 2007/3) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(xy + f(x)) = xf(y) + f(x)$$

for all $x, y \in \mathbb{R}$.

67. (IMO Shortlist 2003) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(f(x) + y) = 2x + f(f(y) - x)$$
for all $x, y \in \mathbb{R}$.

68. (IMO 92/2) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(x^2 + f(y)) = y + (f(x))^2$$
for all $x, y \in \mathbb{R}$.

69. (Irish 95) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$xf(x) - yf(y) = (x - y)f(x + y)$$
for all $x, y \in \mathbb{R}$.

70. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$(x + y)[f(x) - f(y)] = f(x^2) - f(y^2)$$
for all $x, y \in \mathbb{R}$.

71. (Korea 2000/2) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(x^2 - y^2) = (x - y)(f(x) + f(y))$$
for all $x, y \in \mathbb{R}$.

72. (Turkey 2004)Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that
$$f(n) - f(n + f(m)) = m$$
for all $m, n \in \mathbb{Z}$.

73. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(xf(y) + x) = xy + f(x)$$
for all $x, y \in \mathbb{R}$.

74. (Japan 2006) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(x)^2 + 2yf(x) + f(y) = f(y + f(x))$$
for all $x, y \in \mathbb{R}$.

75. (Czech Rep. 2004/6) Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that
$$x^2(f(x) + f(y)) = (x + y)f(yf(x))$$
for all $x, y \in \mathbb{R}^+$.

76. (Czech Rep. 2002/6) Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(xf(y)) = f(xy) + x$$

for all $x, y \in \mathbb{R}^+$.

77. (Czech-Slovak-Polish Match2001/5) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 + y) + f(f(x) - y) = 2f(f(x)) + 2y^2$$

for all $x, y \in \mathbb{R}$.

78. (Iran 2006/6) Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$x^2 f(f(x) + f(y)) = (x + y)f(yf(x))$$

for all $x, y \in \mathbb{R}^+$.

79. (Usamo 2002) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 - y^2) = xf(x) - yf(y)$$

for all $x, y \in \mathbb{R}$.

80. (Belarussia 95) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x + y)) = f(x + y) + f(x)f(y) - xy$$

for all $x, y \in \mathbb{R}$.

81. (India 2005/6) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 + yf(z)) = xf(x) + zf(y)$$

for all $x, y, z \in \mathbb{R}$.

82. (Nordic 2003/4) Determine all the functions $f : \mathbb{R}^* \to \mathbb{R}^*$ such that

$$f(x) + f(y) = f(xyf(x + y))$$

for all $x, y \neq 0$ and $x + y \neq 0$.

83. (Korea 2002/2) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x - f(y)) = f(x) + xf(y) + f(f(y))$$

for all $x, y \in \mathbb{R}$.

84. (Spain 2000/6) Prove that there is no function $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(f(n)) = n + 1$$

for all $n \in \mathbb{N}$.

85. (Spain 98/5) Find all strictly increasing functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(n + f(n)) = 2f(n)$$

for all $n \in \mathbb{N}$.

86. (Canada 2002/5) Find all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that

$$xf(y) + yf(x) = (x + y)f(x^2 + y^2)$$

for all $x, y \in \mathbb{N}_0$.

87. (Romania 2004/3) Find all injective functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(f(n)) \le \frac{n + f(n)}{2}$$

for all $n \in \mathbb{N}$.

88. (Romania TST 92/1) Suppose that $f : \mathbb{N} \to \mathbb{N}$ is an increasing function such that $f(f(n)) = 3n$ for all $n \in \mathbb{N}$. Find $f(1992)$.

89. (IMO 2008/4) Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\frac{(f(w))^2 + (f(x))^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$

for all positive real numbers x, y, z, w satisfying $wx = yz$.

90. (Slovenia 97) Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that for all $m \in \mathbb{Z}$:

$$f(f(m)) = m + 1.$$

91. (Austria 89) Find all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that for all $n \in \mathbb{N}_0$:

$$f(f(n)) + f(n) = 2n + 6.$$

92. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$:

$$f(f(f(n))) + f(f(n)) + f(n) = 3n.$$

93. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$:

$$f(f(n)) + f(n) = 6n$$

94. Find all strictly increasing functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$:

$$f(f(n)) = 3n.$$

95. Find all functions $f : \mathbb{Z} - \{0\} \to \mathbb{Q}$ such that for all $x, y \in \mathbb{Z} - \{0\}$:

$$f(\frac{x+y}{3}) = \frac{f(x)+f(y)}{2}.$$

96. (Poland 2008/3)Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real x, y:

$$f(f(x) - y) = f(x) + f(f(y) - f(-x)) + x.$$

97. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real x, y,

$$f(x+y) - f(x-y) = f(x)f(y).$$

98. Find all functions $f : \mathbb{Q}^+ \mapsto \mathbb{Q}^+$ such that for all $x, y \in \mathbb{Q}^+$,

$$f(x) + f(y) + 2xyf(xy) = \frac{f(xy)}{f(x+y)}.$$

99. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real x, y,

$$xf(y) - yf(x) = f\left(\frac{y}{x}\right).$$

100. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\forall x, y \in \mathbb{R}^+$:

$$f(f(x) + x + y) = xf\left(1 + xf\left(\frac{1}{x+y}\right)\right).$$

101. Find all $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$:

$$f\big(f(x) - f(y)\big) = (x-y)^2 \cdot f(x+y).$$

102. (Hong-Kong 99/4)Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$:

$$f(x + yf(x)) = f(x) + xf(y).$$

103. Find all function $f : \mathbb{Z} \to \mathbb{Z}$ such that for all $x, y \in \mathbb{Z}$:

$$f(x + y + f(y)) \;=\; f(x) + 2y.$$

104. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$:

$$f(x^2 + y + f(y)) = (f(x))^2 + 2y.$$

105. (Vietnam TST 2004)Find all real values of a, for which there exists one and only one function $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f(x^2 + y + f(y)) = (f(x))^2 + ay$$

for all $x, y \in \mathbb{R}$.

106. (Turkey 2005/1)Find all $f:[0,\infty)\to[0,\infty)$ such that for all $x\geq 0$:
    (a) $4f(x)\geq 3x$,
    (b) $f(4f(x)-3x)=x$.
107. (India TST 2001/3)Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that for all $x>0$:
$$f(f(x)-x)=2x.$$
108. Find all $f:\mathbb{R}\to\mathbb{R}$ such that for all $x,y\in\mathbb{R}$:
$$f\big(f(x)-y^2\big)=(f(x))^2-2f(x)y^2+f(f(y)).$$
109. Find all $f:\mathbb{R}\to\mathbb{R}$ such that for all $x,y\in\mathbb{R}$:
$$f\big(x-f(y)\big)=4f(x)-f(y)-4x.$$
110. Find all $a\in\mathbb{R}$ for which the functional equation $f:\mathbb{R}\to\mathbb{R}$
$$f\big(x-f(y)\big)=a\big(f(x)-x\big)-f(y)$$
has a unique solution.
111. (Brazil 2006)Find all functions $f\colon\mathbb{R}\to\mathbb{R}$ such that for every reals x,y:
$$f\big(xf(y)+f(x)\big)=2f(x)+xy.$$
112. (Germany 2008)Determine all functions $f:\mathbb{R}\mapsto\mathbb{R}$ with $x,y\in\mathbb{R}$ such that
$$f(x-f(y))=f(x+y)+f(y).$$
113. (Germany 2007)Determine all functions $f:\mathbb{R}^+\mapsto\mathbb{R}^+$ which satisfy
$$f\left(\frac{f(x)}{yf(x)+1}\right)=\frac{x}{xf(y)+1}\quad\forall x,y>0.$$
114. (Germany 2006)Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ which have the property:
$$f\left(x\right)f\left(y\right)=2f\left(x+yf\left(x\right)\right)$$
for all positive real numbers x and y.
115. (Germany 2006)Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that
$$f\left(x+y\right)+f\left(x\right)f\left(y\right)=f\left(xy\right)+2xy+1$$
for all real numbers x and y.
116. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $\forall x,y\in\mathbb{R}$:
$$f(xf(y))+f\big(f(x)+f(y)\big)=yf(x)+f(x+f(y)).$$